Prelim S13: Quantum Mechanics

Time 120 minutes

Select 2 of the following 3 problems:

Problem 1: Bound States (total: 20pts):

Consider a one-dimensional particle in a finite energy well, with the potential energy

$$V(x) = \begin{cases} 0, & x < -a \\ -2V_0 & -a \le x < 0 \\ -V_0 & 0 \le x < a \\ 0 & x \ge a \end{cases} \quad for$$

where V_0 and $2V_0$ describe the depth of potential well. Answer the following questions.

1. What is the range of energy of the bound states for a particle in this potential? (3pts)
2. For a particle with mass *m* in a bound state with energy E_n, its wave function can be written in the form of $\psi_n(x) = Ae^{\kappa x}$ for x < -a. Derive κ as a function of E_n and m and write down the form of the wave function for x > a. (8pts)
3. What's the form of the wave function in the regimes -a ≤ x < 0 and 0 ≤ x < a respectively? (3pts)
4. Write down the two boundary conditions at x = 0 using the forms of the wave functions. (6pts)

Problem 2: Harmonic Oscillator (total: 20pts)

A. Show that

$$\varphi(x) = \alpha\left(2x^2 - 1\right)e^{\left(-\frac{x^2}{2}\right)}; \quad x = q\sqrt{\frac{m\omega}{\text{h}}}$$

is an eigenfunction of the harmonic oscillator and calculate the corresponding eigenenergy. (7pts)

B. A particle of mass *m* moves in the oscillator potential $V(q)=m\omega^2 q^2/2$. Determine the probability for finding the particle outside the classically allowed region, if the particle is in the ground state of the harmonic oscillator, $\varphi_0(x) = c_0 e^{-x^2/2}$. (7pts)

C. A photon of energy 2ε=7.2eV is being emitted due to the linear harmonic oscillation of an atom (mass m = $4.85\cdot10^{-23}$g) in a molecule. This is being interpreted as a transition from the lowest excited state of the harmonic oscillator to the ground state.
   1. What is the classical amplitude, *A*, of the oscillation of the atom after the transition? (3pts)
   2. What is the probability of finding the atom a distance farther than *A* away from its equilibrium position? Assume the oscillation (3pts)

Useful numbers and relations:

$\hbar=1.055\cdot10^{-34}$ Js

1 eV=$1.6019\cdot10^{-19}$ Js

$$\frac{1}{\sqrt{\pi}}\int_{-1}^{+1} e^{-x^2}\,dx = erf(1) = 0.8427$$

Problem 3: Potentials (total: 20pts):

A. The wave function of a particle of mass m is

$$\psi(\vec{r},t) = \frac{1}{(\pi b^2)^{1/4}} \exp\left(-\frac{r^2}{2b^2} - i\frac{\hbar}{2mb^2} t \right),$$

where $b = \sqrt{\hbar / m\omega}$ is a constant with the dimension of a length, and ω is a fixed frequency. Determine the potential energy, $V(\vec{r})$, of the particle. Describe your result. (8pts)

B. The wave function of a particle moving in a double δ-potential,

$$V(x) = -V_0\delta(x+x_0) - V_0\delta(x-x_0); \qquad V_0>0,$$

Can be written as

$$\psi(x) = c_\pm \left(e^{-\kappa|x-x_0|} \pm e^{-\kappa|x+x_0|} \right),$$

with

$$\kappa = \frac{m}{\hbar^2} V_0 \left(1 \pm e^{-2\kappa x_0} \right) \left(= \sqrt{-\frac{2mE}{\hbar^2}} \right).$$

The $c_\pm$ are determined by normalization. By discussing the expression

$$f(\kappa) = \kappa \left(\frac{\hbar^2}{mV_0} \right) - 1 = \pm e^{-2\kappa x_0}$$

graphically answer the following questions:

1. What is the minimum and what the maximum number of solutions? Specify them in terms of their symmetry? (4pts)
2. Describe how the possibility of obtaining these solutions changes as function of V_0 and x_0? (4pts)
3. Sketch the eigenenergies of the solutions as a function of x_0, and discuss the limiting cases $x_0=0$ and $x_0 \rightarrow \infty$. (4pts)