Question 1

A homogeneous circular disk (radius $R$, mass $M$) has an additional point-like mass $m=M/2$ located at its rim. The disk rolls without friction, and without gliding on a horizontal line while experiencing gravitation.

1. Calculate the coordinates $x_M, y_M$ of the center of the disk as function of the angle $\varphi$. Reference in a way that $q=0$ when $x_M=0$. (1pt)

2. Calculate the coordinates of the point-like mass $x_m, y_m$, as well as the coordinates, $x_{CM}$ and $y_{CM}$, of the center of mass of the combined system of disk and point-like mass as a function of $\varphi$. (3pts)

3. Calculate the kinetic energy, $T(\varphi, \frac{d\varphi}{dt})$, and the potential energy $V(\varphi)$. (4pts)

4. Construct the Lagrange function $L(\varphi, \frac{d\varphi}{dt})$ and the corresponding equation of motion for $\varphi$. (4pts)

5. Calculate the constraining force on the disk caused by the horizontal line. (4pts)

6. Because of the additional point-like mass, a large enough initial velocity $v=dx_M/dt$ can cause the disk to “lift off” the horizontal line. How large does $v$ have to be for that to happen when $\varphi=2\pi/3$? (4pts)
Question 2

A clever way of visualizing motion is through the concept of phase space. For example, one can visualize the motion of a simple harmonic oscillator by representing position on the x-axis and acceleration on the y-axis.

1) What does the phase space diagram for a simple harmonic oscillator without damping look like? (make a 2D sketch, without scaling)

2) What does the phase space diagram for a very weakly damped harmonic oscillator look like? (make a 2D sketch, without scaling)

3) Write the solution to a simple harmonic oscillator (undamped).

4) What equation describes the undamped oscillating object's trajectory in phase space?

5) Consider a particle of mass \( m \) subject to a force of strength \( +kx \), where \( x \) is the displacement of the particle from equilibrium. Calculate the phase space trajectories of the particle.

(20pts)
Question 3 (Classical Mechanics; 20 points)

A circular wire hoop of radius $R$ rotates about a vertical axis passing through its diameter. The hoop rotates with a fixed angular velocity $\omega$. A bead of mass $m$ is threaded on the wire hoop and moves along the hoop without friction. Denote the position of the bead on the hoop by the angle $\theta$, measured with respect to the bottom of the hoop. Denote the acceleration due to gravity by $g$.

(i) Write down the Lagrangian $L(\theta, \dot{\theta})$ for this system in terms of the generalized coordinate $\theta$.

(ii) Find all equilibrium positions $\theta_0$, i.e. those values of $\theta$ for which the bead’s position relative to the hoop does not change (assuming $\dot{\theta} = 0$ initially). You should find that there is a critical rotation speed $\omega_c$, above which there are more equilibria than below. What is $\omega_c$, expressed in terms of $R$, $m$, and $g$?