Question 1. Consider two spin-1/2 particles $\vec{s}_1$ and $\vec{s}_2$. Each spin has two eigenstates $\chi_+ = |\uparrow\rangle$ (spin up state) and $\chi_- = |\downarrow\rangle$ (spin down state). In matrix form, we denote
\[
\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
The Pauli matrices for each spin are then
\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
respectively. (a) Assume a state of these two spins is $|\psi\rangle = \frac{1}{\sqrt{3}} |\uparrow\rangle_1 |\uparrow\rangle_2 + \frac{1}{\sqrt{6}} |\uparrow\rangle_1 |\downarrow\rangle_2 + \frac{1}{\sqrt{2}} |\downarrow\rangle_1 |\downarrow\rangle_2$, where the indices 1 and 2 refer to spin 1 and spin 2, respectively. Please compute the average of the Pauli matrix $\sigma_z$ of the first spin in this state. (b) Please compute the average of product operator $\sigma_z \sigma_{z2}$ in this state. (c) Assume that the Hamiltonian of this system is
\[
H = \hbar \omega_1 \sigma_{z1} + \hbar \omega_2 \sigma_{z2},
\]
where $\omega_1$ and $\omega_2$ are the angular frequencies of the spins. Please write down the expression for the state $|\psi(t)\rangle$ when $|\psi(t = 0)\rangle = |\psi\rangle$.

Question 2. Consider a particle with mass $m$ in a potential energy
\[
V(x) = \begin{cases} 
0, & \text{if } x < -a \\
-V_0, & \text{if } a > x \geq -a \\
0, & \text{if } x \geq a
\end{cases}
\]
where $V_0 > 0$ is the height of potential well. (a) Write down the Hamiltonian of the particle in the regime of $x \geq a$. (2) Assume that the particle has an eigenenergy $E < 0$. Derive the form of this eigenstate in all regimes. (3) Derive the boundary conditions at $x = a$ for this state.

Question 3. The angular momentum for a 3-dimensional particle is defined as $\vec{L} = \vec{r} \times \vec{p}$. It can shown that the $x$-component of the angular momentum is $L_x = y p_z - z p_y$, the $y$-component is $L_y = z p_x - x p_z$, and the $z$-component is $x p_y - y p_x$, where $p_\alpha = -i \hbar \frac{\partial}{\partial x_\alpha}$ with $x_\alpha = x, y, z$, respectively. We also denote $L^2 = \vec{L} \cdot \vec{L} = L_x^2 + L_y^2 + L_z^2$. (a) Please show that $\vec{L} \times \vec{L} = i \hbar \vec{L}$. (b) Derive the commutation relation between $L^2$ and $L_x$. (3) For the angular momentum states $|j, m\rangle$ with integer $j$ and $m = -j, -j + 1, -j + 2, \ldots, j$, $L^2 |j, m\rangle = \hbar^2 j(j + 1) |j, m\rangle$ and $L_z |j, m\rangle = m |j, m\rangle$. Please compute the average $\langle jm|L_x|jm\rangle$ using the commutation relations.