UC Merced E&M Prelim

2016

Choose 2 out of 3 problems. **Be sure to indicate which ones you chose.**

For reference, here are the basic equations of electromagnetism in SI units:

$$\vec{\nabla} \cdot \vec{\mathbf{E}} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \tag{2}$$

$$\vec{\nabla} \cdot \vec{\mathbf{B}} = 0 \tag{3}$$

$$\vec{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial_t} \tag{4}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{5}$$

$$U = \frac{1}{2} \int dV \left[\epsilon_0 |E|^2 + \frac{1}{\mu_0} |B|^2 \right] \tag{6}$$

1. **(20 pts.)** Imagine an infinitely long cylinder of radius R filled with uniform charge density ρ. The axis of the cylinder is along the z-axis. You may assume the charge is fixed to the cylinder.

   Please use cylindrical coordinates $s/\phi/z$ for all answers, where s is the cylindrical radial coordinate (sometimes called ρ, here we use s to distinguish it from charge density). Also, when describing fields do not forget to include direction!

   (a) **(5 pts.)** What is the electric field everywhere in space?

   (b) **(5 pts.)** If we move the cylinder at a velocity $\vec{\mathbf{v}} = +v\hat{\mathbf{z}}$ (along it's axis), what is the magnetic field everywhere in space? Assume we are in the non-relativistic limit, $v \ll c$.

   (c) **(10 pts.)** What happens if $v \sim c$? In this limit we need to be very precise about how we define the charge density, so let us consider two cases:

      i. ρ is measured from the frame of the stationary observer (in which the field is being measured)

      ii. ρ is measured from the rest frame of the cylinder, or, equivalently, we took a stationary rigid cylinder with fixed charge density ρ and then accelerated it to v. (In this case the density is measured in a different frame than the field!)

      For each case, would this change the answers to (a) and (b)? Explain why or why not, and if it changes compute by how much.

2. **(20 pts.)** Consider the electric field of a travelling wave in free space (no charges, no dielectric or diamagnetic materials):

   $$\vec{\mathbf{E}} = E_0 \; \cos\left(kz - \omega t\right) \hat{\mathbf{y}}$$

   (a) **(5 pts.)** What is $\vec{\mathbf{B}}$?

   (b) **(5 pts.)** What is k in terms of ω?

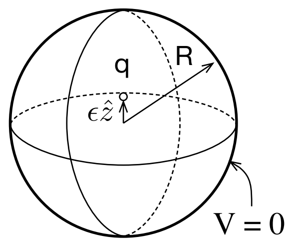
   (c) **(5 pts.)** What is the *average* energy density of the wave? How is this split between the electric and magnetic fields?

(d) **(5 pts.)** If the wave were in a linear dielectric material with permittivity $\epsilon_r = \epsilon/\epsilon_0 > 1$, how would this change the answer to (a-c)? Please give quantitative expressions, and assume $\vec{\mathbf{E}}$ remains unchanged. (Hint: the second of Maxwell's equations remains unchanged in a material, and the fourth becomes $\vec{\nabla} \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f + \frac{\partial \vec{\mathbf{D}}}{\partial_t}$.)

3. **(20 pts.)** Consider a **single** charge, $+q$, offset from the origin by some small amount, $\vec{\mathbf{r}}' = \epsilon\ \hat{\mathbf{z}}$.

(a) **(5 pts.)** The potential from this charge, V_q, can be approximated by a multipole expansion about the origin. Compute the first two terms in this expansion as a function of r and θ (spherical coordinates), i.e. include the monopole (ϵ^0) and dipole (ϵ^1) terms, but ignore quadrupole (ϵ^2) and higher contributions.

(b) **(5 pts.)** Now imagine placing this charge in a conducting sphere of radius R with $V = 0$, centered on the origin so that the charge is (slightly) offset.



Solve for the total potential inside the sphere – both from the charge and the conducting sphere – up to the dipole term computed above.

Note: the solution to Laplace's equation ($\nabla^2 V = 0$) in spherical coordinates is (ignoring ϕ terms):

$$V = \sum_{\ell=0}^{\infty} \left[Ar^{\ell} + Br^{-(\ell+1)} \right] P_{\ell}(cos\theta)$$

$$\begin{aligned} P_0(x) &= 1 & \qquad P_4(x) &= \tfrac{35x^4 - 30x^2 + 3}{8} \\ P_1(x) &= x & \qquad P_3(x) &= \tfrac{5x^3 - 3x}{2} \\ P_2(x) &= \tfrac{3x^2 - 1}{2} & \qquad & \ldots \end{aligned}$$

(c) **(5 pts.)** What is surface charge on the inside surface of the conductor, up to and including the dipole-induced term?

(d) **(5 pts.)** What is the force on the charge, up to and including the dipole-induced term? (Hint: separate V into parts which come from the charge and parts that come from the conducting sphere – which parts produce a force on the charge?)