Fall 2021 Physics Preliminary Exam: Quantum Mechanics

Please pick two problems to complete. On your answer sheet, please clearly indicate which two problems you pick and do not forget to put your name.

**Problem 1: Quantum particle in a box**

Consider a quantum particle with mass *m* confined in a one-dimensional (1D) box potential with V=0 if $0 \leq x \leq L$ and V=∞ otherwise. There is no y- or z- degrees of freedom. [The formulas may be useful:

$\sin(2x) = 2 * \sin(x) * \cos(x)$ ; $\cos(2x) = 1 - 2 * \sin^2(x)$; $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ ; $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$.]

(1a) Please find the wavefunctions of the ground state (the state with the lowest energy) and the first excited state (the state with the second lowest energy). You need to clearly verify the normalization conditions.

(1b) Using the results from (1a), please show that the ground state is orthogonal to the first excited state.

(1c) If the particle is in the ground state, where do we have the highest probability of observing the particle?

(1d) Please find the expectation value of the momentum operator *p* with respect to the ground state.

(1e) Please find the variance the momentum operator *p* with respect to the ground state.

(1f) Without explicitly evaluating the variance of the position operator *x*, please find an upper limit of the variance of *x* by using the result of (1e). [Hint: The uncertainty principle may help.]

**Problem 2: Spin-1/2 system**

Consider a spin-1/2 system that can be represented by a two-dimensional Hilbert space. That means a state is a two-component column vector and an operator is a 2x2 matrix.

(2a) Given two states s1 and s2 represented by $\begin{pmatrix} i/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$ and $\begin{pmatrix} -1/\sqrt{5} \\ 2i/\sqrt{5} \end{pmatrix}$, respectively, please find the probability of observing the system in state s2 if it is prepared in state s1. Here $i^2 = -1$.

(2b) Take state s1 represented by $\begin{pmatrix} i/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$. What happens after the operator $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ acts on state s1?

(2c) Again, take state s1 represented by $\begin{pmatrix} i/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$, please find the expectation value of the operator $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ with respect to s1.

(2d) Please find the eigenvalues and normalized eigenvectors of the operator $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

**Problem 3: Hydrogen atom**

The wavefunction of an electron in a hydrogen atom has the form $\psi_{nlm}(r, \theta, \phi)$, ignoring the spin degrees of freedom. Here *n, l*, and *m* are the principal quantum number, the total angular momentum index, and the z-component angular momentum index.

(3a) The eigen-energy of level *n* is given by $-E_0/n^2$, where $E_0$ is the ionization energy. What is the energy difference between the two states $\psi_{3,2,0}(r, \theta, \phi)$ and $\psi_{3,1,1}(r, \theta, \phi)$?

(3b) Let's consider states with *l=0* and *m=0* (the s-states), so the wavefunctions depend on *r* only. If we prepare an electron in the superposition state $\Psi = (1/\sqrt{10})[\psi_{100}(r) + 2\,\psi_{200(r)} + \sqrt{5}\,\psi_{300}(r)]$, what is the probability of observing the electron with *n=2*?

(3c) Following (3b), if the electron is prepared in state $\Psi$ and then observed with *n=2*, what is the probability of observing the electron with *n=3* immediately after the first measurement?

(3d) Given $\psi_{1,0,0}(r) = \frac{1}{\sqrt{\pi}} * \left(a_0^{-\frac{3}{2}}\right) * \exp(-r/a_0)$, please find the probability of finding the electron within a distance *R* from the center. Here $a_0$ is the Bohr radius.