# Aug. 2021 CM Prelim

## UC Merced

**Attempt only 2 out of the 3 problems**

1. A certain rocket carries a fraction $\alpha$ of its initial mass ($m_o$) as fuel. (That is, the mass of the fuel is $\alpha m_o$.)

   (a) What is the rocket's final speed, accelerating from rest in free space, if it burns all its fuel in a single stage? Express your answer as a multiple of exhausting speed $v_{\text{ex}}$.

   (b) Suppose instead it burns the fuel in two stages as follows: In the first stage it burns a mass of $\frac{\alpha}{2}m_o$ of fuel. It then jettisons the first-stage fuel tank, which has a mass of $\beta m_o$, and then burns the remaining $\frac{\alpha}{2}m_o$ of fuel. Find the final speed in this case, assuming the same value of $v_{\text{ex}}$ throughout.

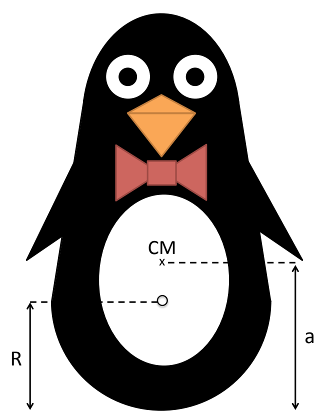
2. Look at the figure below, which shows a possible design for a child's toy. The designer (you!) wants to build a "weeble" (which wobbles, but doesn't fall down). It is a single, solid object, the base shape is a perfect hemisphere (radius R) on the bottom, connected to the rest of the body (here, a penguin). The CM is shown, it is $a$ above the base. You don't need to try to compute $a$, assume it is a given quantity! (As the designer, how might you control/change the position of $a$?)

   (a) Write down the gravitational potential energy when the weeble is tipped an angle $\theta$ from the vertical, as a function of given quantities ($m$, $a$, $R$, $g$, and of course $\theta$).

   (b) Assuming the toy is released from rest, determine the condition(s) for any equilibrium point(s). Then consider stability: what values of (or relations between) $a$ or $R$ ensure that the weeble doesn't fall down? Explain. Can this toy work as desired?

   (c) Write down the Lagrangian of the toy with the following simplifications: the toy rolls without slip and it has only a point-like mass $m$ at CM.

   (d) Solve the Euler-Lagrange equation for small $\theta$ to get the oscillation frequency of the toy.



3. The path of light can be formulated as an optimization problem (for minimal travel time), which is also known as the Fermat's principle. The speed of light is $c/n$, where $c$ is the speed of light in vacuum and $n$ the refractive index. Given two points $(x_1, y_1)$ and $(x_2, y_2)$.

   (a) Formulate the travel time of light as an integral form with a spatial dependent refractive index $n(x) = 1/x$.

   (b) Write down the Euler-Lagrange Equation of the above integral.

   (c) Solve the Euler-Lagrange Equation to find the path of light $y(x)$. Sketch this path if the light travels between points $(1, 2)$ and $(2, 1)$.