Aug. 2020 EM Prelim

UC Merced

Answer 2 of the 3 problems below. A sheet of general EM and vector calculus equations is also attached (mostly the front/back inside covers of Griffiths).

1. Consider two cylindrical coaxial shells of charge, with radius $s = R$ and $s = 2R$. (All charge in this problem lies on the surface of these cylinders.) The inner shell has charge per unit length $+\lambda$ and the outer shell has charge per unit length $-\lambda$, so that the total charge is 0.

   For your answers below, use a cylindrical coordinate system: s, ϕ, z with unit vectors $\hat{\mathbf{s}}$, $\hat{\phi}$, $\hat{\mathbf{z}}$. (Note: here $s = \sqrt{x^2 + y^2}$ is the cylindrical radial coordinate. Often this is denoted with ρ, but here I follow the convention in Griffiths to avoid confusion with charge density.)

   Also, make sure that answers which are vectors are indicated properly.

   (a) What is the electric field, $\vec{\mathbf{E}}$ everywhere in space?

   (b) Using the electric field, find the total energy per unit length required to assemble this charge distribution. ('Per unit length' refers to length along the axis of the cylinder, $\hat{\mathbf{z}}$.)

   (c) What is the electric potential, V, everywhere in space? Make sure that $V(s \to \infty) = 0$.

   (d) The potential field is the energy per unit charge. Thus, one might expect we can find the total energy using:

   $$U_V \stackrel{?}{=} \int \rho(\vec{\mathbf{x}}) V(\vec{\mathbf{x}})\ d^3x \qquad \text{(volume charges)} \qquad (1)$$

   $$= \sum Q_i V_i \qquad \text{(discrete charges)} \qquad (2)$$

   $$= \int V\ dQ \qquad \text{(generalized charge distribution)} \qquad (3)$$

   where ρ is a *volume* charge density.

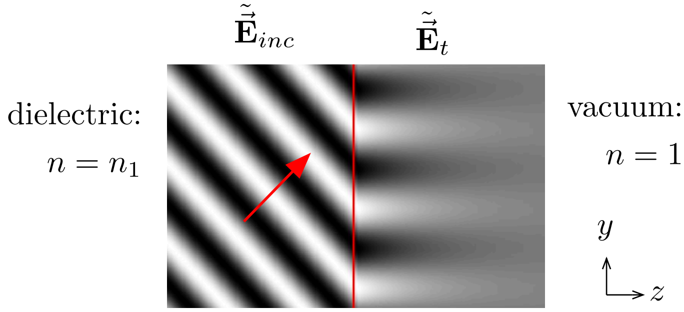
   Compute this energy (per unit length) for the cylindrical charge configuration, using your result from (c).

   (e) If you did everything correctly, your answers in (b) and (d) should be off by a factor of two. Why? (I'm not asking for a formula; explain in words why these two energies are not the same thing.)

2. When light travels from vacuum into a dielectric material it is *partially* reflected. On the other hand, when light is leaving a dielectric material, it can be 100% reflected if the incoming angle is too steep, a phenomena known as total internal reflection.

   It turns out that boundary conditions prevent the electric field from going immediately to 0 in the vacuum region, and so instead we get a wave which exponentially decays in one direction.

$$\tilde{\vec{\mathbf{E}}}_t = E_0 e^{-\kappa z} e^{i(k_y y - \omega t)}\ \hat{\mathbf{x}} \tag{4}$$



   Where κ is a *real* exponential decay constant, and this wave is assumed to be in vacuum ($n = 1$). Note that here we are working with complex fields, and (as always) the *real* field is given by the real part. (However, please leave your answers in the complex form!)

   (a) Using the vacuum wave equation in 3D (below), find κ as a function of ω, c, and k_y.

$$\frac{\partial^2 \tilde{\vec{\mathbf{E}}}}{\partial t^2} = c^2 \nabla^2 \tilde{\vec{\mathbf{E}}} \tag{5}$$

   (Hint: $\nabla^2 (f\hat{\mathbf{x}}) = (\nabla^2 f)\hat{\mathbf{x}}$.)

   (b) Find $\tilde{\vec{\mathbf{B}}}_t$ from Maxwell's equations and the $\tilde{\vec{\mathbf{E}}}_t$ given above. You may assume that $\tilde{\vec{\mathbf{B}}}_t \propto e^{-i\omega t}$.

   (c) The time-averaged Poynting vector of an oscillating field is given by:

$$\left\langle \vec{\mathbf{S}} \right\rangle = \frac{1}{2\mu_0} \mathrm{Re}\left[\tilde{\vec{\mathbf{E}}} \times \tilde{\vec{\mathbf{B}}}^* \right] \tag{6}$$

   Compute $\left\langle \vec{\mathbf{S}} \right\rangle$ for $\tilde{\vec{\mathbf{E}}}_t$.

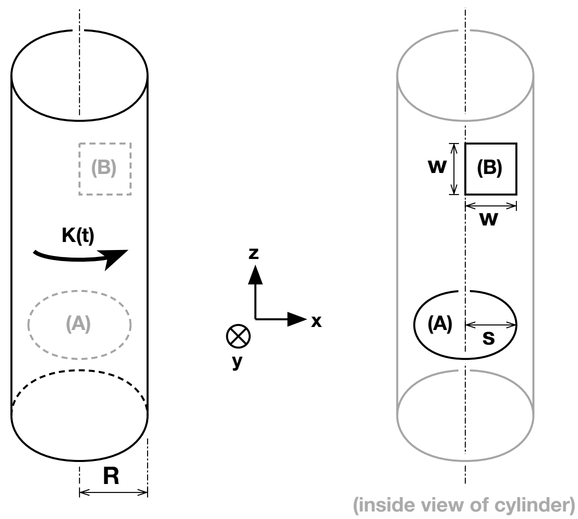
   (d) What is the total energy per unit area being transported from the dielectric ($z < 0$) into the vacuum region ($z > 0$)?

3. Suppose we have an infinite cylinder of radius R with a time-varying current flowing around the outside:

$$\vec{\mathbf{K}} = K(t)\ \hat{\phi} \tag{7}$$

$$\vec{\mathbf{B}}_{inside}(t) \cong \mu_0 K(t)\ \hat{\mathbf{z}} \tag{8}$$

where it is assumed that $K(t)$ is changing slowly enough that the static result is approximately valid.



(a) Find the magnetic flux, Φ_A and Φ_B, through each of the two loops shown above. Both loops are totally inside the cylinder. (A) is a circle of radius s lying in the x-y plane, and (B) is a square of side w lying in the x-z plane (so that the normal is in the $\hat{\mathbf{y}}$ direction) with one edge touching the z-axis.

(b) Now let us assume the current is increasing slowly in a quadratic fashion:

$$K(t) = \alpha t^2 \tag{9}$$

Where α is a constant, assumed to be small enough that the static results for $\vec{\mathbf{B}}$ is still valid. What is the induced EMF, $\mathcal{E}_A$ and $\mathcal{E}_B$, through each loop using the fluxes obtained above?

(c) Determine the electric field, $\vec{\mathbf{E}}$, inside the cylinder using your answer in part (b). Don't worry about the field outside the cylinder.

(d) You should have found in part (c) that your electric field is changing in time. As a result, there is also a displacement current, $\vec{\mathbf{J}}_d$. Find the value of this displacement current (magnitude *and* direction) inside the cylinder assuming your answer from (c) is still valid.