Please choose 2 (and only 2) to answer. On your answer sheet please clearly state which two problems you choose.

Question 1. Consider two spin-1/2 particles $\vec{s}_1$and $\vec{s}_2$. Each spin has two eigenstates $\chi_+ = |\uparrow\rangle$ (spin up state) and $\chi_- = |\downarrow\rangle$ (spin down state). In matrix form, we denote

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The Pauli matrices for each spin are then

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

respectively. (a) Assume a state of these two spins is $|\psi\rangle = \frac{1}{\sqrt{3}}|\uparrow_1, \uparrow_2\rangle + \frac{1}{\sqrt{6}}|\uparrow_1, \downarrow_2\rangle + \frac{1}{\sqrt{2}}|\downarrow_1 \downarrow_2\rangle$, where the indices 1 and 2 refer spin 1 and spin 2, respectively. Please compute the average of the Pauli matrix σ_{z1} of the first spin in this state. (b) Please compute the average of product operator $\sigma_{z1}\sigma_{z2}$ in this state. (c) Assume that the Hamiltonian of this system is

$$H = \hbar\omega_1\sigma_{z1} + \hbar\omega_2\sigma_{z2},$$

where ω_1 and ω_2 are the angular frequencies of the spins. Please write down the expression for the state $|\psi(t)\rangle$when $|\psi(t = 0)\rangle = |\psi\rangle$.

Question 2. Consider a particle with mass m in a potential energy

$$V(x) = \{ \begin{array}{ll} 0, & x < -a \\ -V_0 & a > x \geq -a \\ 0 & x \geq a \end{array}$$

where $V_0 > 0$ is the height of potential well. (a) Write down the Hamiltonian of the particle in the regime of $x \geq a$. (2) Assume that the particle has an eigenenergy $E < 0$. Derive the form of this eigenstate in all regimes. (3) Derive the boundary conditions at $x = a$ for this state.

Question 3. The angular momentum for a 3-dimensional particle is defined as $\vec{L} = \vec{r} \times \vec{p}$. It can shown that the x-component of the angular momentum is $L_x = yp_z - zp_y$, the y-component is $L_y = zp_x - xp_z$, and the z-component is $xp_y - yp_x$, where $p_\alpha = -i\hbar\frac{\partial}{\partial x_\alpha}$ with $x_\alpha = x, y, z$, respectively. We also denote $L^2 = \vec{L}\cdot\vec{L} = L_x^2 + L_y^2 + L_z^2$. (a) Please show that $\vec{L} \times \vec{L} = i\hbar\vec{L}$. (b) Derive the commutation relation between L^2 and L_x. (3) For the angular momentum states $|j, m\rangle$ with integer j and $m = -j, -j+1, -j+2, \cdots, j-1, j$, $L^2|j, m\rangle = \hbar^2 j(j+1)$ and $L_z|j, m\rangle = m|j, m\rangle$. Please compute the average $\langle jm|L_x|jm\rangle$using the commutation relations.