1. A particle of mass m is in a one-dimensional harmonic oscillator potential of frequency ω, whose Hamiltonian is

$$\hat{H} = \frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{x})^2\right] = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \tfrac{1}{2}\right),$$

where $\hat{a}_+$, $\hat{a}_-$ are respectively the raising and lowering operators.

(a) By applying the raising operator $\hat{a}_+ = \dfrac{1}{\sqrt{2\hbar m\omega}}\left(-i\hat{p} + m\omega\hat{x}\right)$ to the ground state wavefunction

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2},$$

derive the wavefunction $\psi_1(x)$ of the first excited energy eigenstate.

(b) Show explicitly that the result from part (a) is a solution of the time-independent Schrödinger equation.

2. An electron in a hydrogen atom is, at $t = 0$, in a state described by the normalized wavefunction

$$\psi(\vec{r}, 0) = A\left(4\psi_{100}(\vec{r}) + 3\psi_{211}(\vec{r}) - \psi_{210}(\vec{r}) + \sqrt{10}\psi_{21-1}(\vec{r})\right)$$

where $\psi_{n\ell m}$ are the usual normalized energy eigenfunctions of the H atom.

(a) Find the normalization factor A.

(b) What is the expectation value of L^2 (the operator for the squared orbital angular momentum) at $t = 0$?

(c) What is the expectation value of the operator $B = -i(L_xL_yL_z - L_yL_xL_z)$ at $t = 0$?

(d) Is $\psi(\vec{r}, 0)$ an energy eigenfunction? Why or why not?

(e) Assuming that no measurement takes place, find the wavefunction $\psi(\vec{r}, t)$ at all times.

3. A spin 1/2 particle is in a state described by the normalized spinor $\chi = \dfrac{1}{\sqrt{65}}\begin{pmatrix} 4 \\ 7i \end{pmatrix}$ where i is the unit imaginary number. (Recall that a spinor $\chi = \binom{a}{b}$ is a linear combination $a\chi_+ + b\chi_-$ of the spin-up state $\chi_+ = |\uparrow\rangle = \binom{1}{0}$ and the spin-down state $\chi_- = |\downarrow\rangle = \binom{0}{1}$, both of which are eigenstates of S_z.)

(a) If you measured S_y of this particle, what values could you get, and what is the probability of each?

(b) What is the expectation value of S_y for this particle?

(c) Suppose you measure S_y and obtain the largest possible value. If you subsequently measure S_y a second time, what values could you get, and what is the probability of each?

(d) Suppose you measure S_y and obtain the largest possible value. If you subsequently measure S_z, what values could you get, and what is the probability of each?

For reference:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Formulas that might be helpful

$$
\begin{aligned}
J_\pm &= J_x \pm i J_y, \\
J_- \left| j\, m \right\rangle &= \hbar \sqrt{(j+m)(j-m+1)} \left| j\, m-1 \right\rangle, \\
J_+ \left| j\, m \right\rangle &= \hbar \sqrt{(j-m)(j+m+1)} \left| j\, m+1 \right\rangle.
\end{aligned}
$$

$$
\begin{aligned}
L_x &= i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi} \right) \\
L_y &= i\hbar \left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi} \right) \\
L_z &= -i\hbar \frac{\partial}{\partial\phi}
\end{aligned}
$$

$$
\begin{aligned}
[x, p_x] &= i\hbar \\
[L_x, L_y] &= i\hbar L_z \qquad \text{and cyclic permutations}
\end{aligned}
$$